

# Confinement and screening of the Schwinger model on the Poincare half plane

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## Abstract

We discuss the confining features of the Schwinger model on the Poincare half plane. We show that despite the fact that the expectation value of the large Wilson loop of massless Schwinger model displays the perimeter behavior, the system can be in confining phase due to the singularity of the metric at horizontal axis. It is also shown that in the quenched Schwinger model, the area dependence of the Wilson loop, in contrast to the flat case, is not a sign of confinement and the model has a finite energy even for large external charges separation. The presence of dynamical fermions can not modify the screening or the confining behavior of the system. Finally we show that in the massive Schwinger model, the system is again in screening phase. The zero curvature limit of the solutions is also discussed.

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# 1 Introduction

One of the exactly soluble model in quantum field theory is the quantum electrodynamics of the massless fermions in 1+1 dimensions, which is known as Schwinger model [1]. The massive Schwinger model, describing the electromagnetic interaction of a massive Dirac field, is no longer exactly soluble, however all non-trivial features of the massless model continue to hold for small fermion mass limit [2]. The Schwinger model may serve as a laboratory to study some important features of particle physics, present also in higher dimensional theories, such as screening and quark confinement which are some of the most important problems in particle physics. For example it has been proposed that the infrared behavior of  $\text{QCD}_4$  may be responsible for the confinement of quarks and gluons. But the concept of infrared slavery, i.e. the increase of potential between colored objects with separation, could not be verified to be true using the perturbation methods (because of infrared singularities) and must be studied nonperturbatively. These kinds of calculations can be done in an equivalent two-dimensional model.

As it is well known, the gauge field of the massless Schwinger model can be made massive by the standard Higgs mechanism and the Coulomb force is replaced by a finite range force. Then by introducing two opposite static external charges  $q$  and  $\bar{q}$ , one can see that the potential tends to some constant for large separation of  $q\bar{q}$  pairs, reflecting the screening of these charges by the induced vacuum polarization. On the other hand in the massive Schwinger model, a semiclassical analysis reveals a linear  $q\bar{q}$  potential. In this case, by computing the Wilson loop for widely separated charges, within the framework of Euclidean path integral and mass perturbation (for small masses), one can see that integer external probe charges are completely screened whereas a linearly potential is formed between widely separated non-integer charges [3].

A particular intriguing and interesting case occurs when the two dimensional surface, on which the model is defined, is a curved space-time. (Similar investigations for pure Yang-Mills theories on arbitrary two dimensional compact Riemann surfaces have been done in several papers, see for example [16,20,21].) These models are useful for better understanding the confinement and screening mechanisms in curved space-time and can be viewed as a first step to study these phenomena in the presence of quantum gravity. Moreover, they may have application in string theory and quantum gravity coupled to nonconformal matter (note that the kinetic term of the gauge field spoils the conformal invariance of the theory).

The Schwinger model has been studied on different non flat surfaces, for example on closed Riemann surfaces of genus  $g \geq 2$  [4], on torus [5], and on sphere [6]. Also the Green function of the gauge field of the Schwinger model has been calculated on the Poincare disk in [7]. Moreover, in [22], the authors have considered a  $D$ -dimensional hyperboloid with negative curvature, embedded in  $(D + 1)$ -dimensional Minkowski space, and by considering

the behavior of the gauge and matter fields near the boundary, they have chosen the solutions with suitable behavior. In this way, they have used the negative curvature space-time as a regulator for interacting Euclidean quantum field theories. However, the confinement and screening properties of the Schwinger model have not yet been studied on curved space-time. In [8], where the bosonization procedure of the Schwinger model in curved space has been discussed, it has been mentioned that this model continues to exhibit screening or confinement of the charges associated to the electromagnetic field on conformally flat spaces. As we will show, it is not true at least for the Poincare half plane. In [22], the authors have argued that as the perimeter and the area in hyperboloid space-times are proportional for large loops, one can not simply distinguish between different phases by only considering the Wilson loop dependence on area or perimeter. As we will show, this is right, i.e. by explicit computation of effective static potential between a quark and antiquark, we show that despite the different behavior of the Wilson loop of the Schwinger model in  $e \neq 0$  and  $e = 0$  (the first has perimeter behavior, while the second has area behavior), both have a common phase structure.

In this paper we want to study the confining behavior of the Schwinger model on the Poincare half plane. This is an interesting case because it can illustrate the effects of the boundary *and* the metric of the space-time on the confinement feature of the Schwinger model. Other property of the Poincare half plane is that its metric is independent of one of the coordinates, so one can obtain the static potential of the external charges in terms of the spatial geodesic distance.

The paper is organized as follows. In section 2, following the method used in [3], we obtain an expression for the potential between the external charges by integrating out the fermionic degrees of freedom. We discuss the confining and screening like behaviors of the system and point out the differences of these features with respect to the flat case. We justify our results by calculating the expectation value of the Wilson loop. We also derive the bosonization rules for the Schwinger model on the Poincare half plane. In section 3 we consider the massive Schwinger model. Using the bosonization method and by solving the equations of motion of the gauge and matter fields, we obtain a perturbative expression for the interaction energy of the probe charges.

Note that in this paper we do not consider the nontrivial topologically sectors of the gauge fields.

## 2 Massless Schwinger model on the Poincare half plane and its confining behavior

The Poincare half plane,  $H = \{(x, t), x > 0\}$ , is a non-compact Riemann surface equipped with the metric  $ds^2 = (dx^2 + dt^2)r^2/x^2$  and the symplectic area form  $\sqrt{g}d^2x = (dx \wedge dt)r^2/x^2$ .  $r$  is a scale parameter of the Poincare plane and is related to the scalar curvature by  $R = -2/r^2$ . This space is conformally related to the compact orientable Riemann surface  $\Sigma_g$  with genus  $g \geq 2$ ,  $\Sigma_g = H/G$ , where  $G$  is a discrete subgroup of  $PSL(2, R)$  (the isometry group of  $H$ ). The geodesics of the Poincare half plane are semi-circles centered on the horizontal axis  $t$  (which we take it as the time axis), and straight lines parallel to the vertical axis  $x$ . The geodesic distance between the points  $(x_1, t_1)$  and  $(x_2, t_2)$  on the semi circle is

$$L = r \cosh^{-1} \left[ 1 + \frac{(x_2 - x_1)^2 + (t_2 - t_1)^2}{2x_1 x_2} \right]. \quad (1)$$

For the points  $(x_1, t)$  and  $(x_2, t)$ ,  $x_2 > x_1$ , situated on the straight line the geodesic distance is given by

$$d = r \ln \frac{x_2}{x_1}. \quad (2)$$

The Schwinger model is defined by the action

$$S = \int \sqrt{g} d^2x [-i\bar{\psi} \hat{\gamma}^a e_a^\mu (\partial_\mu - ieA_\mu) \psi + \frac{1}{4} g^{\mu\rho} g^{\nu\lambda} F_{\mu\nu} F_{\rho\lambda}], \quad (3)$$

where  $e$  is the charge of dynamical fermions, and  $\hat{\gamma}^a$  are anti-Hermitian matrices which in terms of Pauli matrices are  $\hat{\gamma}^0 = i\sigma_2$  and  $\hat{\gamma}^1 = i\sigma_1$ .  $F_{\mu\nu}$  is defined by  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ ,  $g_{\mu\nu} = \eta_{\mu\nu} r^2/x^2$ ,  $\eta_{\mu\nu} = \text{diag}(1, 1)$ , and  $\sqrt{g} = r^2/x^2$ . The zweibeins fields  $(e_a^\mu, e_\mu^a)$  are defined through

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}, \quad g^{\mu\nu} = e_a^\mu e_b^\nu \eta^{ab}. \quad (4)$$

For the metric  $g_{\mu\nu} = \eta_{\mu\nu} r^2/x^2$ , we obtain

$$\begin{aligned} e_\mu^a &= \frac{r}{x} \delta_\mu^a, & e_a^\mu &= \frac{x}{r} \delta_a^\mu, \\ e^{\mu a} &= \frac{x}{r} \eta^{\mu a}, & e_{\mu a} &= \frac{r}{x} \eta_{\mu a}. \end{aligned} \quad (5)$$

The action (3) is invariant under change of coordinate system, frame rotation  $e_\mu^a \rightarrow \Lambda_b^a e_\mu^b$ ,  $\Lambda \in SO(2)$ , and local gauge transformation, but it is not conformal invariant since the Maxwell field theory is conformal invariant only in four dimensions.

## 2.1 Bosonization

The classical equation of motion of the field  $A_\mu$  is

$$\frac{1}{\sqrt{g}}\partial_\nu\sqrt{g}F^{\nu\sigma} = J^\sigma = -e\bar{\psi}\hat{\gamma}^b e_b^\sigma \psi,$$

which yields

$$\partial_\sigma\sqrt{g}\bar{\psi}\hat{\gamma}^b e_b^\sigma \psi = 0.$$

Hence

$$\bar{\psi}\hat{\gamma}^b e_b^\sigma \psi = \alpha\epsilon^{\sigma\nu}\partial_\nu\Phi, \quad (6)$$

where  $\alpha$  is a constant,  $\epsilon^{\sigma\nu} = \hat{\epsilon}^{\sigma\nu}/\sqrt{g}$  and  $\hat{\epsilon}^{01} = \hat{\epsilon}_{01} = 1$ ,  $\hat{\epsilon}^{10} = -1$ . This relation is one of the bosonization rules for massless fermions in a two dimensional (conformally flat) space. In [8], it has been shown that by performing a fermionic change of variables,  $\psi = \chi/g^{1/8}$  and  $\bar{\psi} = \bar{\chi}/g^{1/8}$ , the bosonization of the fermionic part of the action (3) is realized in a similar method as in the flat case. On the other hand the bosonization rules on the half plane,  $\mathbb{R}^+ \times \mathbb{R}$ , is the same as the complete plane [9]. Therefore on the Poincare half plane the bosonization rules are [10]

$$\begin{aligned} -i\bar{\psi}\gamma^\mu\partial_\mu\psi &= \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi, \\ \bar{\psi}\gamma^\mu\psi &= \frac{i}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\Phi, \end{aligned} \quad (7)$$

$$\bar{\psi}\psi = \frac{1}{g^{1/4}}\bar{\chi}\chi = -\frac{1}{g^{1/4}}\Sigma\cos(2\sqrt{\pi}\Phi),$$

in which  $\gamma^\mu = \hat{\gamma}^a e_a^\mu$ , and  $\Sigma$  is a  $c$ -number which depends on the normal ordering of the composite operator  $\bar{\psi}\psi$ . To determine  $\Sigma$ , we proceed as [11].

On one hand, the bosonization of the composite operator  $\bar{\chi}\chi$  is the same as in the flat case, that is  $\bar{\chi}\chi = -\Sigma N_\mu \cos(2\sqrt{\pi}\Phi)$ , where  $N_\mu$  is the normal ordering with respect to the mass  $\mu = e/\sqrt{\pi}$ . Hence

$$\begin{aligned} \langle \bar{\chi}\chi(\xi_1)\bar{\chi}\chi(\xi_2) \rangle &= \Sigma^2 \langle N_\mu \cos[2\sqrt{\pi}\Phi(\xi_1)] N_\mu \cos[2\sqrt{\pi}\Phi(\xi_2)] \rangle \\ &= \frac{\Sigma^2 r^2}{x_1 x_2} \cosh[4\pi D(\xi_1, \xi_2)], \end{aligned} \quad (8)$$

where  $\xi_1 = (x_1, t_1)$ ,  $\xi_2 = (x_2, t_2)$ , are two points on the upper half plane, and  $D(\xi_1, \xi_2)$  is the bosonic propagator [7]

$$D(\xi_1, \xi_2) = \frac{1}{2\pi} Q_l \left( 1 + \frac{2|\xi_1 - \xi_2|^2}{4x_1 x_2} \right), \quad (9)$$

computed from the Lagrangian

$$L = \frac{1}{2}g^{\mu\nu}\sqrt{g}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{2}\mu^2\sqrt{g}\Phi^2. \quad (10)$$

$Q_l$  is the Legendre function of the second kind and  $l = (-1 + \sqrt{1 + 4\mu^2 r^2})/2$ . The appearance of the metric dependent term  $r^2/(x_1 x_2)$  in eq.(8) is related to the renormalization of vertex operators on the curved space-time [12]. In the limit  $\xi_1 \rightarrow \xi_2$ , we use the asymptotic behavior of  $Q_l$  and obtain

$$\langle \bar{\chi}\chi(\xi_1)\bar{\chi}\chi(\xi_2) \rangle = \frac{\Sigma^2 r^2}{2x_1 x_2} \exp[-\ln \frac{|\xi_1 - \xi_2|^2}{4x_1 x_2} - 2\gamma - 2\Psi(l+1)], \quad (11)$$

where  $\gamma$  is the Euler constant, and  $\Psi$  is the digamma function.

On the other hand, in the limit  $\xi_1 \rightarrow \xi_2$ , we have [7]

$$\langle \bar{\chi}\chi(\xi_1)\bar{\chi}\chi(\xi_2) \rangle = \frac{1}{2\pi^2 |\xi_1 - \xi_2|^2}. \quad (12)$$

Note that this relation is the same as one in flat space-time. The reason of this equality lies in the fact that in the limit  $\xi_1 \rightarrow \xi_2$ , all the  $A_\mu$ -dependent terms in evaluating  $\langle \bar{\chi}\chi(\xi_1)\bar{\chi}\chi(\xi_2) \rangle$  are canceled out [5], and this calculation reduces to one in the free fermion model, i.e. without gauge field, on a flat Euclidean space-time, described by the action [8]

$$S_{\text{free}} = \int d^2x (-i\bar{\chi}\hat{\gamma}^a \partial_a \chi). \quad (13)$$

Comparing (11) and (12) we obtain

$$\Sigma = \frac{1}{2\pi r} \exp[\gamma + \Psi(l+1)], \quad (14)$$

which differs from the result obtained for the complete flat plane:  $(e/2\pi^{3/2})\exp(\gamma)$  [10]. This difference is due to the presence of the curvature which modify the Green function of the gauge fields appeared in the fermionic two-point functions [5, 7]. In the limit  $R \rightarrow 0$  ( $r \rightarrow \infty$ ), using  $\lim_{x \rightarrow \infty} \Psi(x) = \ln(x)$ , we obtain the same  $\Sigma$  as the flat case.

## 2.2 Confinement: the effective action approach

In order to investigate the confining behavior of the action (3), we will obtain the equation of motion of the gauge field derived from the corresponding effective action. Using (7), the bosonic version of (3) is

$$S = \int \sqrt{g} d^2x \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{ie}{\sqrt{\pi}} \epsilon^{\mu\nu} A_\mu \partial_\nu \Phi + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right). \quad (15)$$

Integrating over the bosonic degrees of freedom we arrive at

$$S_{\text{eff}} = \int \left( -\frac{e^2}{2\pi} \sqrt{g} \frac{F}{\sqrt{g}} \frac{1}{\Delta} \frac{F}{\sqrt{g}} + \frac{1}{2\sqrt{g}} F^2 \right) d^2x, \quad (16)$$

in which  $\Delta = (1/\sqrt{g}) \partial_\mu g^{\mu\nu} \sqrt{g} \partial_\nu$ , and  $F = \hat{\epsilon}^{\mu\nu} \partial_\mu A_\nu$ .

As an alternative method, this effective action can be also obtained by integrating out the fermionic degrees of freedom of the action (3). To do this, we should compute the determinant of the Dirac operator  $i\gamma^\mu D_\mu = i\gamma^\mu(\partial_\mu - ieA_\mu)$ ,

$$D := -\ln \frac{\int D\bar{\psi} D\psi \exp(\int \sqrt{g} d^2x \bar{\psi} i\gamma^\mu (\partial_\mu - ieA_\mu) \psi)}{\int D\bar{\psi} D\psi \exp(\int \sqrt{g} d^2x \bar{\psi} i\gamma^\mu \partial_\mu \psi)} = -\ln \frac{\det i\gamma^\mu D_\mu}{\det i\gamma^\mu \partial_\mu}. \quad (17)$$

Using the one-loop radiative correction of the two-point function of the gauge field and also by considering the requirement of the invariance of the theory under  $\text{PSL}(2, \mathbb{R})$ , it can be shown that [7]

$$D = -\frac{e^2}{2\pi} \int \sqrt{g} d^2x \frac{F}{\sqrt{g}} \frac{1}{\Delta} \frac{F}{\sqrt{g}}. \quad (18)$$

Adding the kinetic term of the gauge field, we arrive at (16).

In the gauge  $A_1 = 0$  and in the static case  $dA_0/dt = 0$ , the effective Lagrangian (16) becomes

$$\mathcal{L}_{\text{eff}} = \frac{e^2}{2\pi} A_0^2 + \frac{1}{2} \frac{x^2}{r^2} \left( \frac{dA_0}{dx} \right)^2. \quad (19)$$

The above effective Lagrangian density shows that the photon gains a mass equal to  $e/\sqrt{\pi}$ , which can be interpreted as a peculiar two-dimensional version of the Higgs phenomenon.

Now following [3], if we introduce a static external charge distribution composed of a quark and an anti-quark with charges  $e_1 = -e'$  and  $e_2 = e'$  at points  $\xi_1 = (a, t)$  and  $\xi_2 = (b, t)$ , respectively, this Lagrangian becomes

$$\mathcal{L} = \frac{e^2}{2\pi} A_0^2 + \frac{1}{2} \frac{x^2}{r^2} \left( \frac{dA_0}{dx} \right)^2 + J^0 A_0 \frac{r^2}{x^2}, \quad (20)$$

where

$$\begin{aligned} J^0 &= \frac{i}{\sqrt{g}} \sum_{n=1}^2 e_n \int \delta^2(\xi - \xi_n) dt_n \\ &= i \frac{x^2}{r^2} e' [\delta(x - b) - \delta(x - a)]. \end{aligned} \quad (21)$$

In this case, the equation of motion of the field  $A_0$  is

$$\frac{d}{dx} \frac{x^2}{r^2} \frac{dA_0}{dx} - \frac{e^2}{\pi} A_0 = ie' [\delta(x - b) - \delta(x - a)]. \quad (22)$$

To find  $A_0(x)$ , we note that the Green function of the self adjoint operator  $P = \frac{d}{dx} \frac{x^2}{r^2} \frac{d}{dx} - \frac{e^2}{\pi}$  is

$$G_P(x, x') = -\frac{r^2}{2l+1} \frac{x_{<}^l}{x_{>}^{l+1}}, \quad (23)$$

where  $x_{<}$  ( $x_{>}$ ) is the smaller (larger) value of  $x$  and  $x'$ . This Green function is the same as the Green function of the radial part of Poisson operator in spherical coordinates and satisfies the Dirichlet boundary condition at  $x = 0$  (the Poincare half plane has no boundary, but by

boundary condition we mean the behavior of the fields near the horizontal axis). In the flat case limit,  $r \rightarrow \infty$ ,  $l$  leads to  $\mu r$  and therefore the eq.(23) reduces to

$$-\frac{(g(x)g(x'))^{1/8}}{2\mu}e^{-\mu d(xx')}, \quad (24)$$

where  $d$  is the geodesic distance (2). By setting  $g_{\mu\nu} = \eta_{\mu\nu}$ , eq.(24) leads to the Green function of the flat case, i.e.  $-(1/2\mu)e^{-\mu d}$ . Using (23), we obtain

$$A_0(x) = ie'[G_P(x-b) - G_P(x-a)] = \begin{cases} -\frac{ie'r^2}{2l+1}\left(\frac{b^l}{x^{l+1}} - \frac{a^l}{x^{l+1}}\right), & b < x \\ -\frac{ie'r^2}{2l+1}\left(\frac{x^l}{b^{l+1}} - \frac{a^l}{x^{l+1}}\right), & a < x < b \\ -\frac{ie'r^2}{2l+1}\left(\frac{x^l}{b^{l+1}} - \frac{x^l}{a^{l+1}}\right), & x < a. \end{cases} \quad (25)$$

To calculate the quark–antiquark energy, we must note that the Schwinger model on the Poincare half plane can be considered as the analytical continuation of the corresponding model on a Minkowskian space–time described by the metric  $ds^2 = (r^2/x^2)(dt^2 - dx^2)$ . By ignoring the  $i$  factors in eqs.(21) and (25) and substituting them back into  $\mathcal{L}^{\text{Min.}}$ , which has the same form as (20), one can obtain the static external charges energy as  $U = -\int \mathcal{L}^{\text{Mink.}} dx = \int \mathcal{L}^{\text{Eucl.}} dx$  [3]. In this way we find the interaction energy of the external charges as

$$U = \frac{1}{2} \int J^0 A_0 \frac{r^2}{x^2} dx = \frac{e'^2}{2l+1} \frac{r^2}{2a} (-2e^{-\frac{d}{r}(l+1)} + e^{-\frac{d}{r}} + 1). \quad (26)$$

For a detailed discussion on the relation of the Euclidean action with the Minkowskian static energy, see [17].

In the flat case, the eq.(22) is replaced by

$$\frac{d^2}{dx^2} A_0 - \frac{e^2}{\pi} A_0 = ie'[\delta(x-b) - \delta(x-a)], \quad (27)$$

which is invariant under translation ( $x \rightarrow x + c, c \in \mathbb{R}$ ), hence the potential is only a function of charge separation, which is a translational invariant quantity. But in our case, (22) is not invariant under scale transformation (dilatation  $x \rightarrow \lambda x; \lambda \in \mathbb{R}$ ), which leaves the distance  $d$  invariant, hence the potential depends on both the distance  $d = r \ln(b/a)$  and the position of the external charges.

For large separation,  $b \gg a$ , the potential tends to

$$\lim_{d \rightarrow \infty} U = \frac{r^2}{2a} \frac{e'^2}{2l+1}, \quad (28)$$

which indicates the *screening* like phenomenon: By fixing the position of one of the charges at an arbitrary point  $x = a$ , and moving the other charge, the potential increases linearly for small separation and tends to a finite value for large  $d$ . But the crucial point is that the geometry of the Poincare half plane is non-trivial, and a model defined in this space–time,



may have different behaviors in different regions. For example, while the confining phase is dominant in a region, the system may be in screening phase in another region. To see this, one must study the behavior of some external charge in this space, as a probe. Now as it is clear from (26), the system is in confining phase near the boundary  $x = 0$ : for  $a \simeq 0$ , we must have  $b = a + O(a^2)$  in order to have a finite energy for the system, otherwise  $U \rightarrow \infty$ . This means that in the massless Schwinger model on the Poincare half plane, the *confining phase*, is dominant *near the horizontal axis*. This is related to the singularity of the metric at  $x = 0$ .

On the flat plane, the Schwinger model is confining in the absence of dynamical fermions: The screening potential [3]

$$U_{\text{flat}} = \frac{e'^2}{2\mu}(1 - e^{-\mu|b-a|}), \quad (29)$$

in the limit  $\mu \rightarrow 0$ , becomes  $(e'^2/2)|b - a|$  which increases with the relative distance of the charges. In this limit, the effects of the fermionic vacuum polarization is switched off: the screening is replaced by the confining behavior of the system.

But on the Poincare half plane at  $\mu = 0$ , that is when dynamical massless fermions are absent, the potential becomes

$$U = \frac{e'^2 r^2}{2a}(1 - e^{-\frac{d}{r}}), \quad (30)$$

which has the same confining or screening nature as (26). The dynamical fermions can only decrease the amount of saturated energy. Hence the screening like (or confining) behavior of the Schwinger model depends on the vacuum polarization *and* the curvature of the space-time.

### 2.3 Confinement: the Wilson loop approach

Now it is interesting to obtain and interpret these results by computing the Wilson loop expectation value. The interaction of an external current density  $j^\mu$  and the gauge field  $A_\mu$  is described by the action  $S_{\text{int.}} = \int \sqrt{g} j^\mu A_\mu d^2x$ . We assume that  $j^\mu$  is produced by two external charges moving on a loop which is obtained as follows. Two charges  $e'$  and  $-e'$  are created at the point  $(x, t)$  and move apart in (Euclidean) time  $\tau$  to points  $(a, t + \tau)$  and  $(b, t + \tau)$ . Then they stay static at their positions for a period of time  $T$ , and after that come together to annihilate. In the limit  $T \gg \tau$ , in which we are interested, this Wilson loop becomes a rectangle  $c$  characterized by  $a, b$ , and  $T$ , on the Poincare half plane. The reason for choosing this kind of Wilson loop is that in the large  $T$ -limit, the expectation value of this Wilson loop becomes proportional to  $\exp[-U(d)T]$  ( $U(d)$  is the static external charge potential) for time-independent metrics [18]. (See also [19] for the same calculations on a curved space-time.) The interaction term of this process is  $S_{\text{int}} = ie' \oint_c dx^\mu A_\mu$ , and the expectation value of the corresponding Wilson loop is

$$< W_c[A] > =$$

$$\frac{\int DA_\alpha D\Phi \delta(H[A_\alpha]) \det[\frac{\delta H[A^\lambda]}{\delta \lambda}] \exp(ie' \oint_c A_\mu dx^\mu) \exp[\int (-\frac{1}{2}(\partial_\mu \Phi)^2 - \frac{ie}{\sqrt{\pi}} F\Phi - \frac{1}{2} \frac{x^2}{r^2} F^2) d^2x]}{\int DA_\alpha D\Phi \delta(H[A_\alpha]) \det[\frac{\delta H[A^\lambda]}{\delta \lambda}] \exp[\int (-\frac{1}{2}(\partial_\mu \Phi)^2 - \frac{ie}{\sqrt{\pi}} F\Phi - \frac{1}{2} \frac{x^2}{r^2} F^2) d^2x]} \quad (31)$$

$H[A_\alpha] = 0$  is the gauge-fixing condition and  $\lambda$  parameterizes the gauge transformation  $A_\alpha^\lambda = A_\alpha + \partial_\alpha \lambda$ . One can show that by using the change of variables  $A \rightarrow (F, \eta)$ ,  $\eta := H[A]$ , the Jacobian of this transformation:

$$DA_\alpha = \det^{-1}[\frac{\delta H[A^\lambda]}{\delta \lambda}] D\eta DF, \quad (32)$$

cancels precisely against the ghost determinant [16]. Thus

$$< W_c[A] > = \frac{\int DFD\Phi \exp[\int (-\frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2} \frac{x^2}{r^2} F^2 - \frac{ie}{\sqrt{\pi}} F\Phi) d^2x] \exp(ie' \oint_c A_\mu dx^\mu)}{\int DFD\Phi \exp[\int (-\frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2} \frac{x^2}{r^2} F^2 - \frac{ie}{\sqrt{\pi}} F\Phi) d^2x]}. \quad (33)$$

Using the Stokes theorem

$$\oint_c A_\mu dx^\mu = \int_D \eta(\xi) F(\xi) d^2x, \quad (34)$$

where  $c = \partial D$ , and  $\eta(\xi) = \begin{cases} 1, & \xi \in D \\ 0, & \xi \notin D. \end{cases}$ , we arrive at

$$< W_c[A] > = \exp[-\frac{e'^2}{2} \int \frac{r^2}{x^2} \eta^2(\xi) d^2\xi - \frac{e^2 e'^2}{2\pi} \int \eta(\xi) G_W(\xi, \xi') \eta(\xi') d^2\xi d^2\xi'], \quad (35)$$

in which the Green function  $G_W(\xi, \xi')$  satisfies

$$[\frac{x^2}{r^2} (\frac{d^2}{dx^2} + \frac{d^2}{dt^2}) \frac{x^2}{r^2} - \mu^2 \frac{x^2}{r^2}] G_W(\xi, \xi') = \delta^2(\xi - \xi'). \quad (36)$$

If we insert the Fourier expansion

$$G_W(\xi, \xi') = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_k(x, x') e^{ik(t-t')} dk, \quad (37)$$

in eq.(36), the coefficients are found as following

$$f_k(x, x') = -r^4 (xx')^{-3/2} I_{l+\frac{1}{2}}(kx_{<}) K_{l+\frac{1}{2}}(kx_{>}). \quad (38)$$

$I_{l+\frac{1}{2}}$  and  $K_{l+\frac{1}{2}}$  are modified Bessel functions of the first and second kind, respectively. Performing the integration over  $k$  we obtain

$$G_W(\xi, \xi') = \frac{r^4}{2\pi (xx')^2} Q_l[\frac{x^2 + x'^2 + (t-t')^2}{2xx'}], \quad (39)$$

Now as the functional integral in the massless Schwinger model is Gaussian, and the higher order correlators factorize into product of pair correlators  $< F(\xi) F(\xi') >$ , we have [3]

$$< \exp(ie' \oint A_\mu dx^\mu) > = \exp[-\frac{e'^2}{2} \int \int_D d^2\xi d^2\xi' < F(\xi) F(\xi') >]. \quad (40)$$

Comparing (40) with (35) and (39), results

$$\langle F(\xi)F(\xi') \rangle = \delta^2(\xi - \xi') \frac{r^2}{x^2} - \frac{\mu^2 r^4}{2\pi (xx')^2} Q_l(\cosh \frac{L}{r}), \quad (41)$$

where  $L$  is the geodesic distance between  $\xi$  and  $\xi'$  (see eq.(1)). By considering the behavior of  $I_{l+\frac{1}{2}}(x)$  at  $x \simeq 0$  ( $Q_l(\cosh \frac{L}{r})$  at  $L = \infty$ ), one can easily check that the behavior of  $f_k(x, x')$  ( $\langle F(\xi)F(\xi') \rangle$ ) is consistent with the Dirichlet boundary condition imposed on eq.(23).

In the flat case limit, one can show that  $\lim_{r \rightarrow \infty} Q_l[\cosh(L/r)] = K_0(\mu L)$ , and by setting  $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ , the eq.(41) becomes

$$\langle F(\xi)F(\xi') \rangle_{\text{flat}} = \delta^2(\xi - \xi') - \frac{\mu^2}{2\pi} K_0(\mu|\xi - \xi'|), \quad (42)$$

which is the strength fields correlator on the flat space-time [13]. In the absence of dynamical fermions ( $\mu = 0$ ), eq.(42) reduces to  $\delta^2(\xi - \xi')$  and one obtains the area law for the Wilson loop, which is a characteristic of a confining potential. In the presence of dynamical fermions, since  $K_0(\mu|\xi - \xi'|)$  decays exponentially as  $e^{-\mu|\xi - \xi'|}$ , the correlator exhibits the finite correlation length (related physically to the screening effect), and the perimeter law is arisen for large contour [14]. On the Poincare half plane,  $K_0$  is replaced by  $Q_l(\cosh \frac{L}{r})$  which decays as  $[\cosh(L/r)]^{-l-1}$  for large  $L/r$ . Hence in this case we have also a finite correlation length for  $\langle F(\xi)F(\xi') \rangle$  and, as we will show, the Wilson loop is perimeter dependent. In fact the area term arising from the delta function is canceled out by the corresponding term in the integration of  $Q_l$ .

Using

$$\lim_{T \rightarrow \infty} \frac{1}{2\pi T} \int_0^T e^{ikt} dt \int_0^T e^{-ikt' dt'} = \delta(k), \quad (43)$$

and

$$f_0(x, x') = -\frac{r^4}{2l+1} \frac{x_{<}^{l-1}}{x_{>}^{l+2}}, \quad (44)$$

one can obtain the following expression

$$U = \lim_{T \rightarrow \infty} (-\frac{1}{T} \ln W) = \frac{r^2}{2l+1} \frac{e'^2}{2} [\frac{1}{a} + \frac{1}{b} - 2(\frac{a^l}{b^{l+1}})], \quad (45)$$

for the static potential between external charges which is equal to one obtained in eq.(26). After some calculations, one can show that the Wilson loop for  $T \gg b \gg a$  is

$$\langle W_c[A] \rangle = \exp[-\frac{r^2}{2l+1} \frac{e'^2}{2} T(\frac{1}{a} + \frac{1}{b}) + O(\frac{1}{T^\lambda})], \quad (46)$$

where using the hypergeometric representation of the Legendre function,  $\lambda$  is found to be  $\lambda = 2l + 1$ .

But the perimeter of the large Wilson loop  $T \gg b, a$  is

$$\oint_c \frac{r\sqrt{dx^2 + dy^2}}{x} = T\left(\frac{r}{a} + \frac{r}{b}\right), \quad (47)$$

therefore eq.(46) shows that for a large contour, the perimeter law is *satisfied*, which is the same behavior as the flat space-time case.

In the quenched Schwinger model ( $\mu = 0$ ), the Wilson loop expectation value is

$$\langle W_c[A] \rangle = \exp\left[-\frac{e'^2}{2} \int \eta^2 \frac{r^2}{x^2} dx dt\right] = \exp\left[-\frac{e'^2}{2} T r^2 \left(\frac{1}{a} - \frac{1}{b}\right)\right]. \quad (48)$$

But we note that  $Tr^2(1/a - 1/b)$  is nothing but the area of the rectangle bounded by the Wilson loop. Therefore on the Poincare half plane, the Wilson loop of the Schwinger model in the absence of dynamical fermions (that is the pure QED<sub>2</sub>) is equal to the exponential of the area, like the flat case. But in contrast to the flat case, the area is not proportional to the geodesic distance of the charges, and we *can not conclude* that the system is in confining phase. This result is consistent with our discussion after eq.(30).

### 3 Confining aspect of massive Schwinger model on the Poincare half plane

The massive Schwinger model, i.e. U(1) gauge theory with massive dynamical fermions of charge  $e$  and mass  $m$ , is defined by the action

$$S = \int \sqrt{g} d^2x [-i\bar{\psi}\hat{\gamma}^a e_a^\mu (\partial_\mu - ieA_\mu)\psi + m\bar{\psi}\psi + \frac{1}{4}g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}]. \quad (49)$$

This model is not soluble even in the flat case, but in the limit  $m \ll e$ , the physical quantities may be evaluated using perturbative expansion in fermions mass.

In conformally flat curved space-time, the bosonic form of the action (49) is [8]

$$S = \int \sqrt{g} d^2x \left[ \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{ie}{\sqrt{\pi}}\epsilon^{\mu\nu}A_\mu\partial_\nu\Phi - \frac{m\Sigma}{g^{\frac{1}{4}}}\cos(2\sqrt{\pi}\Phi) + \frac{1}{4}g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} \right], \quad (50)$$

where the constant  $\Sigma$  is given by (14). In fact this model is a Sine-Gordon model whose interaction is position dependent [8]. The confining behavior of this system can be analyzed perturbatively by expanding the mass term in a power of  $\Phi$  [15]. The equations of motion followed from the bosonized action (50), in the presence of external charges (21), are

$$\begin{aligned} \frac{d}{dx} \frac{x^2}{r^2} \frac{d}{dx} A_0 + \frac{ie}{\sqrt{\pi}} \frac{d\Phi}{dx} &= ie'[\delta(x-b) - \delta(x-a)], \\ -\frac{d^2\Phi}{dx^2} + \frac{ie}{\sqrt{\pi}} \frac{dA_0}{dx} + \frac{2\sqrt{\pi}m\Sigma r}{x} \sin(2\sqrt{\pi}\Phi) &= 0, \end{aligned} \quad (51)$$

where we have used, as before, the Coulomb gauge  $A_1 = 0$  and  $dA_0/dt = 0$ . Using the approximation  $\sin(2\sqrt{\pi}\Phi) \simeq 2\sqrt{\pi}\Phi$ , and assuming that the field  $\Phi$  is a slowly varying field [15], we arrive at

$$\begin{aligned} \frac{d}{dx} \left( \frac{x^2}{r^2} + \frac{e^2}{4\pi^2 m \Sigma r} x \right) \frac{dA_0}{dx} &= ie' [\delta(x-b) - \delta(x-a)], \\ \frac{4\pi m \Sigma r}{x} \Phi + \frac{ie}{\sqrt{\pi}} \frac{dA_0}{dx} &= 0, \end{aligned} \quad (52)$$

with solutions

$$A_0(x) = \begin{cases} 0, & x > b \\ -i \frac{4\pi^2 m \Sigma e' r}{e^2} \left[ \ln\left(1 + \frac{e^2}{4\pi^2 m \Sigma} \frac{r}{b}\right) - \ln\left(1 + \frac{e^2}{4\pi^2 m \Sigma} \frac{r}{x}\right) \right], & a < x < b \\ -i \frac{4\pi^2 m \Sigma e' r}{e^2} \left[ \ln\left(1 + \frac{e^2}{4\pi^2 m \Sigma} \frac{r}{b}\right) - \ln\left(1 + \frac{e^2}{4\pi^2 m \Sigma} \frac{r}{a}\right) \right], & x < a, \end{cases} \quad (53)$$

and

$$\Phi(x) = \begin{cases} 0, & x > b \\ -\left(\frac{e'}{e}\right) \frac{e^2}{4\pi^2 m \Sigma} \left( \frac{1}{\frac{x}{r} + \frac{e^2}{4\pi^2 m \Sigma}} \right), & a < x < b \\ 0, & x < a. \end{cases} \quad (54)$$

Therefore we find the potential  $U = \frac{1}{2} \int \rho A_0 dx$  as

$$U = 2\pi^2 m \Sigma \left( \frac{e'}{e} \right)^2 r \left[ \ln\left(1 + \frac{e^2}{4\pi^2 m \Sigma} \frac{r}{a}\right) - \ln\left(1 + \frac{e^2}{4\pi^2 m \Sigma} \frac{r}{b}\right) \right]. \quad (55)$$

By fixing  $a$  and increasing the separation of the charges,  $U$  increases and finally achieves the limiting value  $2\pi^2 m \Sigma (e'/e)^2 r \ln(1 + \frac{e^2 r}{4\pi^2 m \Sigma a})$ , which shows that the system is in the screening phase. When one of the charges is located near the horizontal axis,  $a \rightarrow 0$ , the eq.(55) goes to infinity, unless  $b \rightarrow a$ . So the system is in confining phase at  $x \simeq 0$ , as we expect. On the other hand, in order to satisfy the conditions that we have assumed for the field  $\Phi$  (to be small and small varying), we must take

$$\left( \frac{x}{r} + \frac{e^2}{4\pi^2 m \Sigma} \right) \gg \left( \frac{e'}{e} \right) \frac{e^2}{4\pi^2 m \Sigma}. \quad (56)$$

By expanding  $U$  in terms of  $r/a$  and  $r/b$ , we obtain

$$U = \frac{e'^2}{2} r^2 \left( \frac{1}{a} - \frac{1}{b} \right) + O\left(\frac{r^2}{a^2}, \frac{r^2}{b^2}\right). \quad (57)$$

For large  $a/r$  and  $b/r$ ,  $U$  is proportional to the area of the Wilson loop characterized by  $a, b$ , and  $T$ . In the flat case, this behavior is interpreted as a sign of confinement but, as we have discussed earlier, this is not true for the Poincare half plane.

Finally if we consider the small fermion mass limit  $m \ll e$ , and also  $e' \ll e$ , the eq.(55) reduces to

$$U = 2\pi^2 m \Sigma \left( \frac{e'}{e} \right)^2 r \ln\left(\frac{b}{a}\right), \quad (58)$$

which is comparable with the corresponding result in the flat case, after substituting  $r \ln(b/a) \rightarrow (b-a)$  and  $\Sigma(\text{Poincare}) \rightarrow \Sigma(\text{flat})$  [3,15]. Note that the potential (58) is proportional to the geodesic distance  $d = r \ln(b/a)$ , but this is not a sign of confinement as in the flat case. To see this, note that if one fixes the position of the first charge at  $x = a$  and moves the other charge to a large distance ( $b/r \rightarrow \infty$ ), the eq.(55) reduces to (for  $m \ll e, e' \ll e$ )

$$U\left(\frac{b}{r} \rightarrow \infty\right) = e'^2 r^2 \frac{1}{2a}, \quad (59)$$

which has a finite value, and the system is again in the screening (and *not* confining) phase.

## 4 Conclusion

Let us summarize the main results of the paper:

1- In  $m = 0$ , the Schwinger model on flat space-time is in screening phase, but on the Poincare half plane, the system is in confining phase in  $x \simeq 0$  region, and in screening phase in regions far enough from the horizontal axis (after eq.(28)),

2- In  $m = 0$  and  $e = 0$ , the model is confining in flat case (after eq.(29)), but on the Poincare half plane, the phase depends on the region under study (after eq.(30)).

3- In  $m = 0$ , the Wilson loop obeys the perimeter (area)-law for  $e \neq 0$  ( $= 0$ ) in both the flat (after eq.(42)) and the Poincare (eqs.(47) and (48)) cases. But on the Poincare half plane, the area dependence does not indicate the confining phase (in contrast to the flat case) (after eq.(48)).

4- In  $m \neq 0$ , the Schwinger model on the flat space-time is in confining phase but in the Poincare case, the model is in screening phase (after eqs.(55) and (59)).

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